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# UNSTEADY INTERACTION OF UNIFORMLY VORTEX FLOWS

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*The problem of the decay of an arbitrary discontinuity (the Riemann problem) for the system of equations describing vortex plane-parallel flows of an ideal incompressible liquid with a free boundary is studied in a long-wave approximation. A class of particular solutions that correspond to flows with piecewise-constant vorticity is considered. Under certain restrictions on the initial data of the problem, it is proved that this class contains self-similar solutions that describe the propagation of strong and weak discontinuities and the simple waves resulting from the nonlinear interaction of the specified vortex flows. An algorithm for determining the type of resulting wave configurations from initial data is proposed. It extends the known approaches of the theory of one-dimensional gas flows to the case of substantially two-dimensional flows.*

**1. Formulation of the Problem.** The system of integrodifferential equations describing vortex liquid flows with a free boundary in the approximation of shallow water theory was studied by Benney [1], Varley and Blythe [2], Freemen [3], and Blythe et al. [4]. Primary attention has been given to the construction of exact solutions — steady flows, simple waves, etc. It has been shown [5, 6] that appropriate generalizations of the notions of the characteristics and hyperbolicity of the system lead to new possibilities for the theoretical analysis of wave motions, based on the generalization of methods of the theory of nonlinear hyperbolic differential equations. In this case, integrodifferential models differ considerably from differential models in that the spectra of the propagation velocities of the characteristics are no longer purely discrete but can contain continuous parts (segments). In this connection, the similarity to the cases that have already been studied is limited, and many elements of the mathematical apparatus should be developed again. A system of relations for strong discontinuities that generalizes the classical model of a hydraulic jump of shallow water theory is proposed and studied by Teshukov [7, 8], who proved the existence of simple waves corresponding to isolated values of the characteristic spectrum and analyzed the main properties of these waves.

The question arises: What types of waves are connected with motions of a general nature? Apparently, waves that correspond only to the discrete region of the characteristic spectrum of the system of equations of motion do not permit one to construct the flow that arises at an arbitrary discontinuity of initial data. An answer to this question can be obtained in studies of the formulation of the problem of the decay of an arbitrary discontinuity for an integrodifferential system.

In the present work, the indicated formulation is analyzed within the framework of the class of exact solutions of the general system that describes liquid flows with piecewise-constant vorticity. This class is chosen, first, to simplify the problem, and second, because of the properties of conservation of vorticity in intersection of fronts of strong discontinuities and zones of simple waves [7, 8].

The system of equations describing the propagation of long waves in a liquid layer $0 \leqslant y \leqslant h(x,t)$ [the equation $y = h(x,t)$ specifies the free boundary] has the form [1, 5]

$$u_t + uu_x + vu_y + gh_x = 0, \quad u_x + v_y = 0, \quad h_t + \Big(\int\limits_0^h u\,dy\Big)_x = 0, \quad v(x,0,t) = 0. \tag{1.1}$$

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Here $x$ and $y$ are Cartesian coordinates on the plane, $t$ is time, $u$ and $v$ are components of the velocity vector in Cartesian coordinates, and $g$ is the acceleration of gravity. The last relation of (1.1) defines the boundary condition of nonpenetration at the bottom.

As has been shown [5, 6], system (1.1) converted to Eulerian–Lagrangian coordinates can be treated as an integrodifferential hyperbolic system of equations having discrete and continuous spectra of characteristic velocities. For the following consideration, it is convenient to formulate the conditions on the characteristics in terms of Euler variables. The corresponding formulas are obtained by simple transformations of the formulas given in [5, 6]. The characteristic curves $x = x(t)$ of system (1.1) are described by the differential equations $dx/dt = k_i(x,t)$, where $k_i$ $(i = 1, 2, \ldots)$ are roots of the secular equation

$$F(k) = g \int_0^h \frac{dy}{(u-k)^2} - 1 = 0. \tag{1.2}$$

These characteristics correspond to the discrete region of the spectrum of characteristic velocities. Along the characteristics, the Riemann invariants are conserved:

$$r_i = k_i - g \int_0^h \frac{dy}{u - k_i}, \qquad r_{it} + k_i r_{ix} = 0. \tag{1.3}$$

The continuous spectrum of characteristic velocities of the system contains the intervals of variation of the function $u$ for fixed $x$ and $t$. To find a characteristic that corresponds to the continuous spectrum, it is necessary to define the function $y(x,\lambda,t)$ as a solution of the problem

$$y_t(x,\lambda,t) + u(x,y(x,\lambda,t),t)y_x(x,y(x,\lambda,t),t) = v(x,y(x,\lambda,t),t), \quad y(x,\lambda,0) = y_0(x,\lambda),$$

and to integrate the ordinary differential equation $dx/dt = k_{(\lambda)}(x,t)$, where $k_{(\lambda)}(x,t) = u(x,y(x,\lambda,t),t)$ ($\lambda$ is a parameter that "numbers" the characteristics). If $y_0(x,\lambda)$ is chosen so that with variation of the parameter $\lambda$, the function $y(x,\lambda,t)$ takes all values from 0 to $h(x,t)$, then for each pair of values of $x$ and $t$, a continuous set of characteristic velocities is determined. Along each of the characteristics of the continuous spectrum, the two Riemann invariants [5] are conserved. If one introduces the notation $\omega = u_y$ (in the long-wave model, this quantity defines vorticity), $\omega(x,y(x,\lambda,t),t)$ is conserved along the characteristic $dx/dt = k_{(\lambda)}(x,t)$ ($\lambda$ is fixed).

System (1.1) admits particular solutions of the form $u = u(y)$, $v = 0$, and $h = \text{const}$ that correspond to steady shear flows in a layer with a horizontal free boundary. A natural generalization of the formulation of the Riemann problem in the case of Eqs. (1.1) is the Cauchy problem

$$(u,h)\Big|_{t=0} = \begin{cases} (u_1(y), h_1) & (x > 0, \quad 0 < y < h_1), \\ (u_2(y), h_2) & (x < 0, \quad 0 < y < h_2), \end{cases} \tag{1.4}$$

where $u_i(y)$ are specified functions and $h_i$ are specified constants [the function $v$ is uniquely defined by the known $u$ and $h$ by virtue of (1.1)]. The solution of problem (1.4) describes unsteady wave processes that occur during interaction of the specified steady shear flows. The formulation of (1.4) simulates a discontinuity of the initial data that can result from the nonlinear evolution of a smooth solution of system (1.1) at $t < 0$. In the present work, we consider special initial data: $u_1(y) = \omega_1 y + u_{01}$ and $u_2(y) = \omega_2 y + u_{02}$ ($\omega_i$ and $u_{0i}$ are constants), which correspond to the interaction of flows with constant vorticities. Since the equations and the boundary conditions are invariant with respect to uniform stretching of the variables $x$ and $t$, we seek a solution of problem (1.4) in the class of self-similar solutions: $u = U(x/t, y)$, $h = H(x/t)$, and $v = t^{-1}V(x/t, y)$.

In what follows, we need some properties of the system of equations that describe two-layer flow with constant vorticity in each of the layers. Let a liquid layer $0 \leqslant y \leqslant h(x,t)$ be divided by the boundary $y = \Delta(x,t)$ into two sublayers (Fig. 1). At $0 \leqslant y \leqslant \Delta(x,t)$, the horizontal component of the liquid velocity has the form $u = \Omega_0 y + u_0(x,t)$, and at $\Delta(x,t) \leqslant y \leqslant h(x,t)$, it has the form $u = \Omega(y - h) + u_1(x,t)$ ($\Omega_0$ and $\Omega$ are specified constants). On the boundary $y = \Delta(x,t)$ the velocity vector is continuous. We introduce $u_2(x,t) = \Omega_0\Delta + u_0(x,t) = \Omega(\Delta - h) + u_1(x,t)$, the horizontal velocity component on the boundary

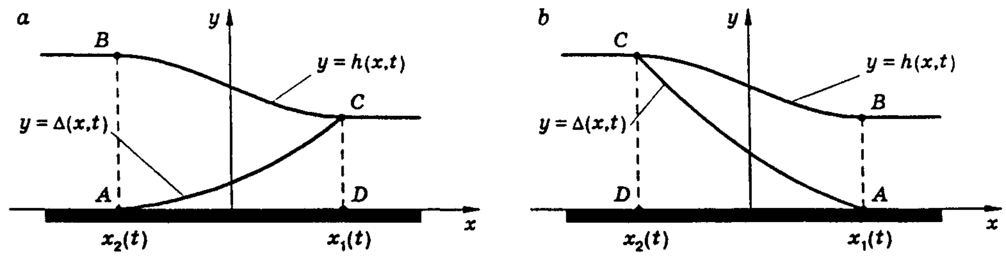


Fig. 1

$y = \Delta(x,t)$. From Eqs. (1.1) for the sought vector $\mathbf{U} = (u_0, u_1, u_2)^t$, where $(\ldots)^t$ is transposition, we obtain the system

$$\mathbf{U}_t + A\mathbf{U}_x = 0, \qquad A = \begin{pmatrix} u_0 - g\Omega_0^{-1} & g\Omega^{-1} & g\Omega_0^{-1} - g\Omega^{-1} \\ -g\Omega_0^{-1} & u_1 + g\Omega^{-1} & g\Omega_0^{-1} - g\Omega^{-1} \\ -g\Omega_0^{-1} & g\Omega^{-1} & u_2 + g\Omega_0^{-1} - g\Omega^{-1} \end{pmatrix}. \tag{1.5}$$

The vertical velocity components in the lower and upper layers are defined, respectively, by the equalities $v = -yu_{0x}$ and $v = (\Delta - y)(u_{2x} - \Omega\Delta_x) - \Delta u_{0x}$, where $\Delta = \Omega_0^{-1}(u_2 - u_1)$ and $h = \Delta + \Omega^{-1}(u_1 - u_2)$.

The secular equation for system (1.5) can be obtained according to the general algorithm of calculating the characteristics, but it is easier to use (1.2):

$$F(k) = \frac{g}{\Omega_0}\frac{u_2 - u_0}{(u_0 - k)(u_2 - k)} + \frac{g}{\Omega}\frac{u_1 - u_2}{(u_1 - k)(u_2 - k)} - 1 = 0. \tag{1.6}$$

In the general case, it has been shown [5] that the secular equation (1.2) has only two roots $k_1(u_0, u_1, u_2)$ and $k_2(u_0, u_1, u_2)$ outside of the range of $u$, $k_1 < u_j$ and $k_2 > u_j$ ($j = 0$, 1, 2). Since the solution of Eq. (1.6) amounts to seeking roots of the third-order polynomial in $k$, Eq. (1.6) has the third real root $k_3(u_0, u_1, u_2)$, which falls within the range of the velocity $u$ in the upper or lower layer.

System (1.5) can be written in the Riemann invariants as

$$r_{it} + k_i r_{ix} = 0 \quad (i = 1,\ 2,\ 3),$$

$$r_i = r_i(u_0, u_1, u_2) = k_i - \frac{g}{\Omega_0}\ln\left|\frac{u_2 - k_i}{u_0 - k_i}\right| - \frac{g}{\Omega}\ln\left|\frac{u_1 - k_i}{u_2 - k_i}\right|.$$

These formulas are obtained from the general formulas (1.3), which give the Riemann invariants of system (1.1).

The family of characteristics of a hyperbolic system is called a strongly nonlinear family [9] if $\nabla k_i \cdot \mathbf{R}_i \neq 0$. Here $\mathbf{R}_i$ is the right eigenvector of the matrix $A$ that corresponds to the eigenvalue $k_i$; $\nabla = (\partial/\partial u_0, \partial/\partial u_1, \partial/\partial u_2)$. It can easily be verified that for the matrix $A$ from (1.5), we have

$$\mathbf{R}_i = ((u_0 - k_i)^{-1}, (u_1 - k_i)^{-1}, (u_2 - k_i)^{-1})^t.$$

Calculation of $\nabla k_i$ using Eq. (1.6) yields

$$\nabla k_i \cdot \mathbf{R}_i = K_i^{-1} L_i,$$

$$L_i = \frac{g}{\Omega_0}\left(\frac{1}{(u_0 - k_i)^3} - \frac{1}{(u_2 - k_i)^3}\right) - \frac{g}{\Omega}\left(\frac{1}{(u_1 - k_i)^3} - \frac{1}{(u_2 - k_i)^3}\right), \tag{1.7}$$

$$K_i = \frac{g}{\Omega_0}\left(\frac{1}{(u_0 - k_i)^2} - \frac{1}{(u_2 - k_i)^2}\right) - \frac{g}{\Omega}\left(\frac{1}{(u_1 - k_i)^2} - \frac{1}{(u_2 - k_i)^2}\right).$$

From this formula it follows that $\nabla k_1 \cdot \mathbf{R}_1 > 0$ and $\nabla k_2 \cdot \mathbf{R}_2 < 0$ [the inequalities $\Omega_0^{-1}(u_2 - u_0) > 0$ and $\Omega^{-1}(u_1 - u_2) > 0$ are used]. Therefore, the characteristic families that correspond to the characteristic roots

$k_1$ and $k_2$ satisfy the conditions of strong nonlinearity. The characteristic family that corresponds to the root $k_3$ of the secular equation is generally not strongly nonlinear. However, if $\alpha = \Omega_0/\Omega$ satisfies the inequalities

$$2^{-1} < \alpha < 2, \tag{1.8}$$

then, in this (and only in this) case, the characteristics $dx/dt = k_3(x,t)$ also form a strongly nonlinear family. Let us show this. We have $K_3 \neq 0$ because $K_3 = F_k(k_3)$, and the functions $F(k)$ and $F_k(k_3)$ can be represented as

$$\begin{aligned} F(k) &= (k-k_1)(k-k_2)(k-k_3)[(u_0-k)(u_1-k)(u_2-k)]^{-1}, \\ F_k(k_3) &= (k_3-k_1)(k_3-k_2)[(u_0-k_3)(u_1-k_3)(u_2-k_3)]^{-1} \neq 0. \end{aligned} \tag{1.9}$$

The last inequality follows from the fact that system (1.5) does not have multiple characteristic roots. The function $L_3$ from (1.7) can be represented as

$$L_3 = (g/\Omega_0)^{-2}\Psi, \quad \Psi = z_0^3 - \alpha z_1^3 + (\alpha-1)z_2^3, \quad z_i = g(\Omega_0(u_i - k_3))^{-1}.$$

The variables $z_j$ are related by the secular equation $\chi = z_0 - \alpha z_1 + (\alpha-1)z_2 - 1 = 0$. The relative position of the points $u = u_i$ and $k = k_3$ on the real axis is given by specifying the quantities $\Omega$ and $\Omega_0$. The following cases are possible:

$$\begin{aligned} &\text{No. 1.} \quad \Omega_0 > \Omega > 0, \quad u_0 < k_3 < u_2 < u_1, \\ &\text{No. 2.} \quad \Omega > \Omega_0 > 0, \quad u_0 < u_2 < k_3 < u_1, \\ &\text{No. 3.} \quad \Omega_0 < \Omega < 0, \quad u_1 < u_2 < k_3 < u_0, \\ &\text{No. 4.} \quad \Omega < \Omega_0 < 0, \quad u_1 < k_3 < u_2 < u_0, \\ &\text{No. 5.} \quad \Omega_0 > 0 > \Omega, \quad u_0 < k_3 < u_1 < u_2, \quad u_1 < k_3 < u_0 < u_2, \\ &\text{No. 6.} \quad \Omega > 0 > \Omega_0, \quad u_2 < u_0 < k_3 < u_1, \quad u_2 < u_1 < k_3 < u_0. \end{aligned} \tag{1.10}$$

Each range of the parameters $u_i$ and $k_3$ (Nos. 1–6) corresponds to a certain range of the variables $z_j$. Thus, for example, $z_0 < 0$ and $z_2 > z_1 > 0$ in case No. 1, $z_1 > 0$ and $0 > z_0 > z_2$ in case No. 2, etc. The function $\Psi$ is tested for a conditional extremum in each of the ranges (with the proviso that $\chi = 0$), and its behavior on the boundaries is then examined. As a result, it is established that $\Psi$ can be of fixed sign only in cases Nos. 1–4 for the parameter $\alpha$ belonging to region (1.8). Next, we assume that the parameters of the problem satisfy condition (1.8). The simple waves of Eqs. (1.5) [solutions in the form $U = U(\beta(x,t))$, where $\beta(x,t)$ is a function of two variables] are defined by the equations

$$(A - kI)U_\beta = 0, \quad \beta_t + k\beta_x = 0,$$

where $k$ satisfies the secular equation (1.6). If one choses the function $k(x,t)$ as $\beta(x,t)$, then $u_j(k)$ ($j = 0, 1, 2$) are found from the first equations, and the last equation serves to determine $k(x,t)$. In the self-similar simple wave, $k(x,t) = x/t$. The first subsystem is easily integrated using the Riemann invariants. Each root of the secular equation $k_i$ corresponds to a solution of the simple-wave type in which $r_j(u_0,u_1,u_2) = \text{const}$ ($j \neq i$). For the vertical velocity component in the simple wave, we have the formulas

$$v = -k_x y u_0'(k), \quad v = k_x((\Delta(k) - y)(u_2'(k) - \Omega\Delta'(k)) - \Delta(k)u_0'(k))$$

for the lower and upper layer, respectively.

**2. Wave of Interaction of Flows.** Let us consider the auxiliary problem of deriving solutions of the simple-wave type that describes the interaction of two vortex flows. We assume that in the regions $x < x_1(t)$ and $x > x_2(t)$ (Fig. 1), the liquid flow is steady and the vorticity is constant ($u_y = \omega_1 = \text{const}$ on the right and $u_y = \omega_2 = \text{const}$ on the left). At $x \in [x_1(t), x_2(t)]$, the contact surface AC $y = \Delta(x,t)$ separates the right region with constant vorticity $\omega_1$ from the left region with constant vorticity $\omega_2$. On the surface AC, conditions of continuity of the velocity vector $(u,v)$ are satisfied and the following differential equation holds:

$$\Delta_t + u(x,\Delta,t)\Delta_x = v(x,\Delta,t).$$

In the cross sections $x = x_i(t)$ ($i = 1, 2$), conditions of continuous joining to the steady flows are satisfied. Further, we shall prove that in the case where $\alpha = \omega_1/\omega_2$ satisfies inequalities (1.8) there is a simple wave — an exact solution of system (1.1) that satisfies the above-mentioned conditions on the boundaries of the region ABCD. Figure 1 shows two possible flow patterns. We determine conditions to which the boundary data on AB and CD should satisfy.

Apparently, the velocity of motion of the point A in the horizontal direction coincides with the horizontal velocity component of the liquid at the bottom [$x_2'(t) = u_A$ in case (a) shown in Fig. 1a, and $x_1'(t) = u_A$ in case (b) (Fig. 1b)]. Similarly, the velocity of motion of the point C in the horizontal direction coincides with $u_C$ [$x_1'(t) = u_C$ in case (a) and $x_2'(t) = u_C$ in case (b)]. In the region ABCD, the flow has a two-layer nature. Hence, to describe its evolution, it is necessary to solve Eqs. (1.5) [for $\Omega_0 = \omega_1$ and $\Omega = \omega_2$ in case (a) or $\Omega_0 = \omega_2$ and $\Omega = \omega_1$ in case (b)]. As the point A is approached, the value $\Delta \to 0$, and, as follows from the secular equation (1.6), the characteristic velocity $k_3$ tends to $u_A$. Similarly, with approach to the boundary CD, the function $h - \Delta \to 0$ and $k_3 \to u_C$. Therefore, the boundaries $x = x_i(t)$ ($i = 1, 2$) of the region ABCD move at characteristic velocity $k_3$. Then, to describe the flow in the interaction region, it is necessary to use a simple wave that corresponds to the characteristics $dx/dt = k_3(x, t)$.

In the simple-wave region, the Riemann invariants $r_1$ and $r_2$ are identically constant and only the invariant $r_3$ changes. We obtain the following necessary condition for the existence of the simple wave of flow interaction:

$$r_1^{(l)} = r_1^{(r)}, \quad r_2^{(l)} = r_2^{(r)}. \tag{2.1}$$

Here $r_1^{(l)}$ and $r_2^{(l)}$ are the values of the Riemann invariants for the left steady flow and $r_1^{(r)}$ and $r_2^{(r)}$ are the values for the right steady flow. In single-layer flow with constant vorticity $\Omega$, the Riemann invariants have the form

$$r_i = k_i - \frac{g}{\Omega} \ln \left| \frac{u_1 - k_i}{u_0 - k_i} \right|, \tag{2.2}$$

where $k_i$ ($i = 1, 2$) are roots of the secular equation

$$\frac{g}{\Omega} \frac{u_1 - u_0}{(u_1 - k)(u_0 - k)} - 1 = 0. \tag{2.3}$$

We note that as $\Delta \to 0$ (or $h - \Delta \to 0$), the Riemann invariants $r_1$ and $r_2$ of the system of equations for two-layer flow gradually becomes $r_1$ and $r_2$ which correspond to the system of equations of single-layer flow. From formulas (2.2) and (2.3) it follows that $r_1 + r_2 = k_1 + k_2 = u_1 + u_0$, and, hence, $u_1^{(r)} - u_0^{(l)} = u_1^{(l)} - u_0^{(r)}$ by virtue of (2.1). Consequently, when equalities (2.1) are satisfied, one of the two inequalities $u_1^{(r)} > u_0^{(l)} mor\ u_0^{(r)} > u_1^{(l)}$ is valid (Fig. 1a and b, respectively). With allowance for the aforesaid, the solution of the simple-wave type $u_0 = u_0(k)$, $u_1 = u_1(k)$, and $u_2 = u_2(k)$ is defined in the region ABCD by the relations

$$r_1(u_0, u_1, u_2) = r_1^{(r)} = r_1^{(l)}, \quad k = k_3(u_0, u_1, u_2), \quad r_2(u_0, u_1, u_2) = r_2^{(r)} = r_2^{(l)}. \tag{2.4}$$

In this case, $k = k_3$ varies from $u_0^{(l)}$ to $u_1^{(r)}$ in case (a) or from $u_0^{(r)}$ to $u_1^{(l)}$ in case (b) (Fig. 1). To determine the qualitative behavior of the flow parameters, we write the following differential equations of the simple wave:

$$(u_0 - k)u_0' + gh' = 0, \quad (u_1 - k)u_1' + gh' = 0, \quad (u_2 - k)u_2' + gh' = 0, \quad gh' = -K_3(k)(L_3(k))^{-1}. \tag{2.5}$$

We analyze the case where $u_1^{(r)} > u_0^{(l)}$ (Fig. 1a) (the case where $u_0^{(r)} > u_1^{(l)}$ is considered in a similar manner). We note that in this case, $\omega_1 > 0$ and $\omega_2 > 0$. Indeed, if $\omega_1 < 0$ and $\omega_2 < 0$, then $u_1^{(r)} < u_0^{(l)}$ and $u_1^{(l)} < u_0^{(l)}$ and, hence, $u_1^{(r)} - u_0^{(l)} < u_0^{(r)} - u_1^{(l)}$. If we take into account that $u_1^{(r)} - u_0^{(l)} = -(u_0^{(r)} - u_1^{(l)})$, it becomes obvious that for $\omega_1 < 0$ and $\omega_2 < 0$ case (b) is realized. Let $\omega_1 > \omega_2 > 0$. Then by virtue of (1.9) and (1.10), $K_3(k) = F_k(k) > 0$ [in (1.10) it is necessary to set $\Omega_0 = \omega_1$, $\Omega = \omega$, and $k_3 = k$]. In the situation considered, $1 < \alpha < 2$, the function $L_3(k)$ is positive in the range of parameters (1.10) (case No. 1) and $h'(k) < 0$, $u_0'(k) < 0$, $u_1'(k) > 0$, and $u_2'(k) > 0$ in the flow region. With approach to the right boundary of the range of definition of the simple wave, $k \to u_1^{(r)}$, and, hence, $u_1 - k \to 0$ and $u_2 - k \to 0$. The secular equation (1.6) defines an integral of system (2.5). From (1.6) it follows that

$$\lim_{u_1-k\to 0} \frac{u_2 - k}{u_1 - k} = (\alpha - 1)\alpha^{-1} \tag{2.6}$$

and the function $h'(k)$ can be represented asymptotically as

$$gh' = -(u_2 - k)\alpha(2\alpha - 1)^{-1} + o(u_2 - k). \tag{2.7}$$

Similarly, as $k \to u_0^{(l)}$, $u_0 - k \to 0$, and $u_2 - k \to 0$, we have

$$\lim_{u_0-k\to 0} \frac{u_2 - k}{u_0 - k} = 1 - \alpha, \quad gh' = -(u_2 - k)(2 - \alpha)^{-1} + o(u_2 - k). \tag{2.8}$$

Hence it follows that the derivatives $u_i'(k)$ have finite limits as $k \to u_1^{(r)}$ and $k \to u_0^{(l)}$:

$$\begin{gathered} u_0'(u_1^{(r)}) = 0, \quad u_1'(u_1^{(r)}) = (\alpha - 1)(2\alpha - 1)^{-1}, \quad u_2'(u_1^{(r)}) = \alpha(2\alpha - 1)^{-1}, \\ u_0'(u_0^{(l)}) = (1 - \alpha)(2 - \alpha)^{-1}, \quad u_1'(u_0^{(l)}) = 0, \quad u_2'(u_0^{(l)}) = (2 - \alpha)^{-1}. \end{gathered} \tag{2.9}$$

In the simple wave considered, the thickness of the lower liquid layer $\Delta(k)$ increases monotonically with increase in $k$, because $\Delta'(k) = \omega_1^{-1}(u_2' - u_0') > 0$. For $k = u_0^{(l)}$, we have $\Delta = 0$, and for $k = u_1^{(r)}$, we obtain $\Delta = h$.

Asymptotic relations (2.6)–(2.8) uniquely define an integral curve of the system of ordinary differential equations (2.5) in a neighborhood of one of the boundary points [$k = u_1^{(r)}$ or $k = u_0^{(l)}$]. This integral curve continues up to the second boundary point. The possibility of continuation follows from the fact that for $k \in (u_0^{(l)}, u_1^{(r)})$, the function $K_3(k)(L_3(k))^{-1}$ does not have singularities ($k \neq u_i$ by virtue of the secular equation). As a result, it is established that system (2.4) [integrals of the differential equations (2.5)] uniquely defines the functions $u_i(k)$ ($i = 0, 1, 2$). In the self-similar problem considered, the relations $u_i = u_i(k)$ and $k = x/t$ give the simple wave of flow interaction.

**3. Particles Trajectories in the Wave of Flow Interaction.** Liquid particles in the unsteady flow considered move along the trajectories $x = x(t)$ and $y = y(t)$:

$$x'(t) = u(x(t), y(t), t), \quad y'(t) = v(x(t), y(t), t). \tag{3.1}$$

For simple waves, $u = u(k, y)$, $v = k_x V(k, y)$, and $k_t + k k_x = 0$. In the plane of the variables $k$ and $y$, we determine curves that correspond to the trajectories (3.1). Differentiating $k(x,t)$ along the trajectory (3.1), we obtain the following differential equations in the plane $k, y$:

$$\bar{k}'(t) = k_t + u k_x = k_x(x(t), t)(u(\bar{k}, y) - \bar{k}), \qquad y'(t) = k_x(x(t), t) V(\bar{k}, y). \tag{3.2}$$

The change of variables $s = \int_{t_0}^{t} k_x(x(\tau), \tau)\, d\tau$ brings system (3.2) to the independent form [$k(s) = \bar{k}(t(s))$]:

$$k'(s) = u(k, y) - k, \quad y'(s) = V(k, y). \tag{3.3}$$

In the case of a centered simple wave, $k = x/t$ and $s = \ln(t/t_0)$. Generally, if any solution $k = k_0(s)$ and $y = y_0(s)$ of system (3.3) is known, to determine the trajectory $x = x(t)$ and $y = y(t)$ it is necessary to solve the following ordinary differential equation for $s(t)$:

$$s'(t) = k_x(x_0(s, t), t).$$

Here the relation $x = x_0(s, t)$ is determined from the equation $k(x, t) = k_0(s)$. Integration gives the functions $x(t) = x_0(s(t), t)$ and $y(t) = y_0(s(t))$ which define the motion of a liquid particle along the trajectory.

Let us analyze the qualitative pattern of integral curves of system (3.3) in the simple wave of flow interaction for case (a) (Fig. 1) ($1 < \alpha < 2$). In the region ACD, $u(k, y) = u_0(k) + \omega_1 y$ and $V(k, y) = -y u_0'(k)$.

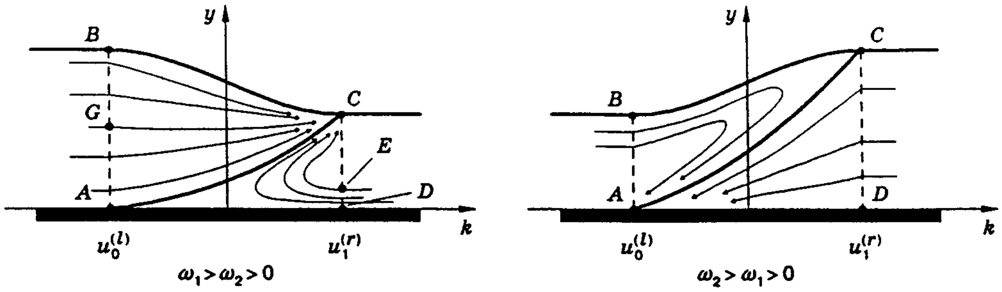


Fig. 2

The singular points $[u(k,y) - k = 0$ and $V(k,y) = 0]$ of the system of ordinary differential equations (3.3) are the points A and C. According to (2.9), $y = 0$ and $u_0(k) = k$ at the point A and $u_1(k) = u_0(k) + \omega_1 h(k) = k$ and $u_0'(k) = 0$ at the point C. For $u_0^{(l)} < k < u_1^{(r)}$, the characteristic root $k_3 = k$ satisfies the inequality $u_0 < k < u_2$, and $V(k,y)$ vanishes only for $y = 0$ $[u_0'(k) < 0]$. Therefore, there are no other singular points in the region ACD. We separate out the linear part of system (3.3) in a neighborhood of the point A, retaining the first terms of the Taylor series of the functions $u(k,y) - k, V(k,y)$:

$$\frac{dk}{ds} = (u_0'(k_A) - 1)(k - k_A) + \omega_1 y = -(2-\alpha)^{-1}(k - k_A) + \omega_1 y,$$

$$\frac{dy}{ds} = -y u_0'(k_A) = y(\alpha - 1)(2-\alpha)^{-1}$$

[relations (2.9) are used]. The eigenvalues of the coefficient matrix of the right side are real and opposite in sign: $\lambda_1 = -(2-\alpha)^{-1}$ and $\lambda_2 = (\alpha-1)(2-\alpha)^{-1}$. Therefore, the singular point A is a saddle. A similar separation of the linear part in a neighborhood of the point C leads to the system

$$\frac{dk}{ds} = (u_0'(k_C) - 1)(k - k_C) + \omega_1(y - h_C) = -(k - k_C) + \omega_1(y - h_C),$$

$$\frac{dy}{ds} = -h_C u_0''(k_C)(k - k_C) = \omega_1^{-1}(1-\alpha)\alpha(2\alpha-1)^{-2}(k - k_C).$$

In the calculation of $u_0''(k_A)$, we use the consequence of Eqs. (2.5) $(u_0(k) - k)u_0'(k) = (u_2(k) - k)u_2'(k)$. Differentiating this equality with respect to $k$ and taking into account that $u_2(k_C) = k_C$, $u_0'(k_C) = 0$ and $u_0(k_C) - k_C = u_0(k_C) - u_1(k_C) = -\omega_1 h_C$, we find $u_0''(k_C) = (\alpha-1)\alpha(2\alpha-1)^{-2}(\omega_1 h_C)^{-1}$. The eigenvalues of the coefficient matrix of the linear part $\lambda_{1,2} = 2^{-1}(-1 \pm (2\alpha-1)^{-1})$ are of identical sign, and, hence, the point C is a node.

The integral curve of Eqs. (3.3) enters the region ACD at the point E of the segment CD; along the curve, $y$ increases monotonically, and $k$ decreases to the turning point, at which the equality $u = k$ holds. Then $k$ begins to increase and the integral curve enters the node C. The pattern of integral curves in the region ACD is shown in Fig. 2.

In the region BCA, the singular points are the points A and C, and the velocity vector $(u, V)$ has the form

$$u(k,y) = u_1(k) + \omega_2(y - h), \qquad V(k,y) = (y - \Delta)(\omega_2 h'(k) - u_1'(k)) - \Delta u_0'(k),$$

where $\Delta(k) = \omega_1^{-1}(u_2(k) - u_0(k))$. At the point A, $y = \Delta = 0$ and $u_2(k_A) = k_A$; at the point C, $y = \Delta = h_C$ and $u_0'(k_C) = 0$. We separate the linear part of system (3.3) in a neighborhood of the point A:

$$\frac{dk}{ds} = -(k - k_A) + \omega_2 y, \qquad \frac{dy}{ds} = \omega_1^{-1}\alpha(\alpha-1)(2-\alpha)^{-2}(k - k_A).$$

The eigenvalues of the coefficient matrix $\lambda_{1,2} = -2^{-1} \pm 2^{-1}\alpha(2-\alpha)^{-1}$ are real and opposite in sign, and, therefore, the point A is a saddle. Separation of the linear part in a neighborhood of the point C leads to the system

$$\frac{dk}{ds} = -\alpha(2\alpha - 1)^{-1}(k - k_C) + \omega_2(y - h_C), \quad \frac{dy}{ds} = (1 - \alpha)(2\alpha - 1)^{-1}(y - h_C).$$

At the point C, the eigenvalues $\lambda_1 = \alpha(2\alpha - 1)^{-1}$ and $\lambda_2 = (\alpha - 1)(2\alpha - 1)^{-1}$ are of identical sign, and, hence, the point C is a node. Along the integral curve that enters the region ABC at the point G of the segment AB, $k$ increases monotonically, and the curve enters the node C. The picture of integral curves in the region ABC is shown in Fig. 2. We note that the integral curves of system (3.3) are not particle trajectories, as follows from the aforesaid. These curves help to understand how particles move relative to the simple wave (the velocity of motion of a point in the horizontal direction along the curve coincides with $u$–$k$). In the simple-wave zone, each straight line $k = k_0 = \text{const}$, $0 \leqslant y \leqslant h$ moves in physical space at velocity $k_0$, $x = k_0 t$. Figure 2 shows patterns of integral curves for different values of vorticity. It is evident that the flow in the region of interaction of the flows has a substantially two-dimensional nature. Along the line of contact of the two swirled flows, jet flow is formed that is directed to the free surface or the bottom, depending on the ratio of the vorticities.

**4. Solution of the Problem of the Decay of an Arbitrary Discontinuity.** Let us determine the configuration of waves that propagate over the specified steady background in the directions $x > 0$ and $x < 0$. The states behind these waves are related by the simple wave of interaction of the flows. Therefore, the wave configuration is determined by condition (2.1), which ensures the existence of the simple wave of interaction of the flows. In the region of single-layer flows, the equations of motion can be written in the form of laws of conservation of mass and momentum:

$$h_t + (u_c h)_x = 0, \qquad (hu_c)_t + (hu_c^2)_x + \left(g\frac{h^2}{2} + \frac{\omega_i^2}{12}h^3\right)_x = 0, \quad i = 1, 2. \tag{4.1}$$

Here $h$ is the depth of the liquid layer, $u_c = 2^{-1}(u_0 + u_1)$, $u_1$ and $u_0$ are the horizontal components of the velocity vector on the free surface and at the bottom, and $\omega_i$ are constants of vorticity. Equations (4.1) are consequences of Eqs. (1.1) for the class of flows with constant vorticity, and they coincide with the equations of gas dynamics if $h$ is identified with the gas density and the pressure $P$ is given by the formula $P = 2^{-1}gh^2 + (12)^{-1}\omega_i^2 h^3$.

The initial formulation of the problem of the decay of an arbitrary discontinuity (1.4) specifies the Cauchy data for Eqs. (4.1):

$$(u_c, h)\Big|_{t=0} = \begin{cases} u_{c1}, h_1, & x < 0, \\ u_{c2}, h_2, & x > 0, \end{cases} \tag{4.2}$$

where $u_{ci}$ and $h_i$ are specified constants. It should be noted that for the class of flows considered, the relations on hydraulic jumps and simple waves that follow from Eqs. (1.1) [7, 8] coincide with the relations on gas-dynamic shock and simple waves. The equation of state of the "gas" satisfies monotonicity and convexity conditions [10], and, therefore, a solution of problem (4.1) and (4.2) can be derived as a combination of a right wave (a simple wave or a hydraulic jump) and a left wave. The difference from the gas-dynamic problem is that the states behind the waves that have passed satisfy the conditions of coincidence of the Riemann invariants $r_1$ and $r_2$ (2.1) rather than the conditions of coincidence of velocities and pressures. By virtue of the relations for a strong discontinuity:

$$[h(u_c - D)] = 0, \quad [hu_c(u_c - D) + P] = 0 \tag{4.3}$$

($D$ is the velocity of the front of the discontinuity and $[f]$ is the symbol of the jump of the functions $f$), the flow parameters $u_c$ and $P$ behind the wave front are related to the flow parameters $u_{c0}$ and $P_0$ ahead of the front by the equation

$$u_c - u_{c0} = \pm\sqrt{(P - P_0)(h_0^{-1} - h^{-1})}. \tag{4.4}$$

For a right wave, $D > u_c$ and the plus sign is chosen in (4.4); for a left wave, $D < u_c$ and the minus sign is chosen. For the subsequent consideration, it is convenient to reduce Eq. (4.4) to the relation between the Riemann invariants behind the wave front $r_1$ and $r_2$ [formula (2.2)] and the Riemann invariants ahead of the

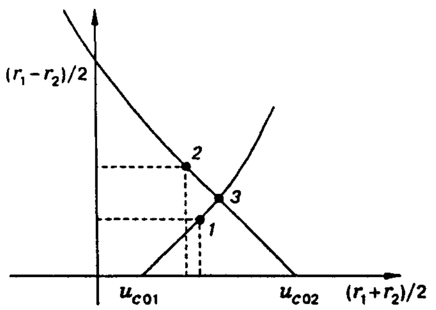


Fig. 3

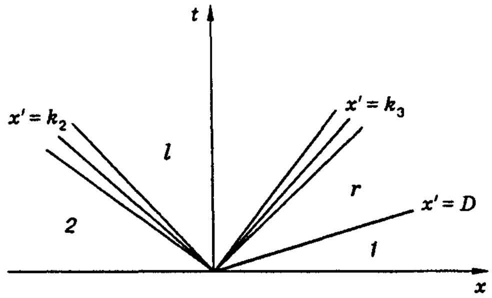


Fig. 4

front $r_{10}$ and $r_{20}$. In this case, we use the equalities

$$u_c = \frac{r_1 + r_2}{2}, \quad h = \frac{u_1 - u_0}{\omega_i}, \quad \frac{r_1 - r_2}{2} = a - \frac{g}{\omega_i} \ln \frac{u_1 - u_0 - 2a}{u_1 - u_0 + 2a},$$
$$a = \sqrt{4^{-1}(u_1 - u_0)^2 + g\omega_i^{-1}(u_1 - u_0)},$$

which follow from the definition of the Riemann invariants $r_i$ (2.2). As a result, Eq. (4.4) takes the form

$$\frac{r_1 + r_2 - r_{10} - r_{20}}{2} = \pm f\Big(\frac{r_1 - r_0}{2}, \frac{r_{10} - r_{20}}{2}\Big), \tag{4.5}$$

where $f(z, z_0)$ increases monotonically on $z$, $f \to \infty$ as $z \to \infty$, and $f(z_0, z_0) = 0$. In the plane of the variables $(r_1 + r_2)/2$ and $(r_1 - r_2)/2$, Eq. (4.5) defines the curve of possible shock transitions from the initial state $r_{10}$, $r_{20}$. The curves of transitions performed by passage of simple rarefaction waves over the given state $r_{10}, r_{20}$ are given by the relations

$$r_2 = r_{20}, \qquad r_1 = r_{10}. \tag{4.6}$$

The first relation describes transitions by means of right simple waves (the level lines of a simple wave move at characteristic velocity $k_1$). The second relation describes transitions for left simple waves (the simple-wave zone moves at characteristic velocity $k_2$). Following the algorithm of solving the gas-dynamic problem of the decay of an arbitrary discontinuity through the initial point $r_{10} = r_1(u_{c1}, h_1)$, $r_{20} = r_2(u_{c1}, h_1)$, we plot the curve of right transitions. For $r_1 - r_2 \geqslant r_{10} - r_{20}$, this curve is described by Eq. (4.5), where the plus is fixed, and for $0 \leqslant r_1 - r_2 \leqslant r_{10} - r_{20}$, it is described by the first equation of (4.6). Similarly, through the point $r_{10} = r_1(u_{c2}, h_2)$, $r_{20} = r_2(u_{c2}, h_2)$ we plot the curve of left transitions. For $r_1 - r_2 \geqslant r_{10} - r_{20}$, this curve is described by Eq. (4.5) (the minus sign is fixed), and for $r_1 - r_2 \leqslant r_{10} - r_{20}$, it is described by the second equation of (4.6). The point of intersection of the curves corresponds to the sought states behind the fronts of waves that have passed. Figure 3 shows diagrams of transitions from the right (1) and left states (2) to state 3 for one of the possible cases of discontinuity decay. As follows from the diagrams, the solution of the problem contains a centered wave propagating over background 2 and a hydraulic jump propagating over state 1. For states $r$ and $l$ (Fig. 4) behind the wave fronts, existence conditions (2.1) for a centered wave of flow interaction are satisfied, and the solution of the initial problem of the decay of an arbitrary discontinuity is completed by constructing the indicated wave in the region $u_0^{(l)} < x/t < u_1^{(r)}$. When the positions of points 1 and 2 change, other wave configurations can arise. In the case where the curves do not intersect ($u_{c02} < u_{c01}$), discontinuity decay can result in two centered waves, behind which one the channel becomes dry ($h = 0$). Here we see an analogy with gas dynamics (flow in vacuum). Figure 4 shows the pattern of the characteristics and fronts of strong discontinuities in the plane $x, t$ for the wave configurations given in Fig. 3.

We note that the analogy with the gas-dynamic problem is not full here. The strong-discontinuity conditions (4.3) guarantee that the average velocity $u_c^{(r)} = 2^{-1}(u_1^{(r)} + u_0^{(r)}) < D$, but the maximum velocity behind the front (in the case considered, where $\omega_i > 0$, it is reached on the free surface) $u_1^{(r)}$ does not necessarily satisfy this inequality. Using the consequence of relations (4.3):

$$(u_c^{(r)} - D)^2 = h_1/h^{(r)}(P^{(r)} - P_1)(h^{(r)} - h_1)^{-1},$$

(the values with the superscript $r$ correspond to the state behind the front, and $h_1$ is the depth ahead of the front) one can reduce the equality $u_1^{(r)} = D$ to the equation

$$C(\xi + 1)(\xi - 1)^{-1}(3\xi^2 + 2\xi + 1)^{-1} = \mathrm{Fr}^2, \quad \xi = h^{(r)}h_1^{-1}. \tag{4.7}$$

Here Fr is a Froude number that characterizes the shift of velocity ahead of the jump [$\mathrm{Fr}^2 = (u_1 - u_0)^2(gh_1)^{-1}$]. The function on the left side of relation (4.7) decreases monotonically with increase in $\xi > 1$, and as $\xi \to 1$, it tends to infinity. Therefore, for the given Fr there is a unique root $\xi_k(\mathrm{Fr})$ of Eq. (4.7). As $\mathrm{Fr} \to \infty$, we have $\xi_k(\mathrm{Fr}) \to 1$, and as $\mathrm{Fr} \to 0$, we have $\xi_k(\mathrm{Fr}) \to \infty$. If the quantity $\xi$, which characterizes the amplitude of the jump, satisfies the inequality $1 < \xi < \xi_k(\mathrm{Fr})$, then $u_1^{(r)} < D$ and the region of flow interaction moves at a velocity lower than the velocity of the discontinuity front. Then, the picture shown in Fig. 4 is possible. For $\xi = \xi_k(\mathrm{Fr})$, the region of flow interaction catches up with the discontinuity front. A further increase in the amplitude of the jump should be accompanied by rearrangement of the wave configuration. This case is not considered in the present paper. Therefore, when configurations with hydraulic jumps arise, the results will refer to the case where the vorticity is low (the $\mathrm{Fr}^2$ is small) and states 1 and 2 are close. Then, the value of $\xi_k(\mathrm{Fr})$ is large, and for small amplitudes of discontinuities, equality (4.7) is not attained. In the case of configurations with centered waves, these limitations do not arise. The analysis performed shows that for small amplitudes of the initial discontinuity and low vorticities, the solution of the problem of the decay of an arbitrary discontinuity contains a left wave, a right wave, and a centered wave of flow interaction. The case of arbitrary amplitudes and the case where inequalities (1.8) are violated require additional studies. In these cases, one might expect occurrence of additional singularities in the region of interaction of the flows.

**Remark.** In a simple wave of flow interaction, the flow velocity at a certain depth coincides with the wave velocity. This flow can be called a simple wave with a critical layer [2]. What distinguishes the point $u = k_3$ from the other values of the velocity that form the continuous characteristic spectrum of system (1.1)? It turns out that at the indicated point, the secular equation (1.2) holds. However, since the integral in (1.2) diverges for $k = u$, the secular equation holds in the sense of the principal value. To verify this fact, it is necessary to consider the analytical continuation $F(z)$ of the characteristic function ($z$ is a complex variable). After integration by parts in (1.2), the integrand has a first-order singularity. This allows one to calculate the limiting values $F^{\pm}(z)$ for $z \to u = k_3$. As a result, we obtain the secular equation (1.6) for $k = k_3$, which is equivalent to the relation $F^{\pm}(k_3) = 0$. The analysis performed suggests that for vortex-flow equations in which the vorticity is not a piecewise-constant function, the solution of the general problem of discontinuity decay yield similar wave configurations.

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